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CRITICAL DAMPING IN CERTAIN LINEAR CONTINUOUS DYNAMIC SYSTEMS

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Abstract—Free damped vibrations of linear elastic structures composed of uniform beam elements with a continuous distribution of mass are studied. Axial, torsional and flexural vibrations are considered. The amount of damping, which can be either internal or external viscous type, varies among the various beam elements of the structure resulting in many critical damping possibilities. A general method is developed which, with the aid of dynamic stiffness influence coefficients defined for every element, determines the "critical damping surfaces" of the system. These surfaces represent the loci of combinations of amounts of damping leading to critically damped motion and thus separating regions of partial or complete underdamping from those of overdamping. The dimension of a critical damping surface is equal to the number of independent amounts of damping present in the system, while the number of these surfaces is infinite, i.e. equal to the number of degrees of freedom of the system. Three examples are presented in detail to illustrate the proposed method for determining critical damping and demonstrate its importance.

1. INTRODUCTUON

The importance of damping as a means of reducing the response of a vibrating structural system is well known. Conventionally, the amount of damping in a linear structural system is expressed as a percentage of the critical damping or modal critical damping values depending on whether damping is everywhere the same in the structure of varies modally, respectively. Thus, it is possible to estimate directly the amount of damping in the structure and to characterize that structure as underdamped, overdamped or critically damped. This knowledge consequently helps one to control the response by appropriate changes of the damping in the structure.

However, for linear structural systems with different viscous damping values for some or all of their members, the problem of determining critical damping becomes much more difficult, because many critical damping possibilities arise. This problem is of considerable importance because the availability of different damping values for different members of a structure provides a more rational way of representing damping properties and permits more effective response control by taking advantage of the freedom of varying the damping of a large number of elements.

The problem of critical damping is part of the general problem of structural free damped vibration, which is concerned with the determination of natural frequencies and modal shapes of viscously damped linear systems. Necessary and sufficient conditions under which discrete and continuous damped linear dynamic systems possess classical normal modes have been established by Caughey and O'Kelly[1]. In a recent paper, Beskos and Boley[2], studied free viscously damped vibrations of linear discrete systems in which the amount of damping varied among the various structural members, thus resulting in many "critical damping surfaces." These surfaces represent the loci of combinations of amounts of damping leading to critically damped motion and thus separate regions of partial or complete underdamping from those of overdamping. A general method for the determination of critical damping surfaces of linear discrete systems was developed in[2]. That method is extended in this paper to certain continuous linear dynamic structural systems. These include one, two or three dimensional structures consisting of uniform beam elements with a continuous distribution of mass, undergoing flexural, torsional or axial free motion, with either internal viscoelastic or external viscous damping. The method developed in[2] is applied to these systems in conjunction with the use of a new kind of dynamic stiffness influence coefficients defined for the aforementioned motions (flexural, torsional and axial) on the basis of the exact solution of the equation of free damped motion of a beam element. Thus, the dynamic problem is reduced to a static-like one and the exact solution of the problem is obtained. The use and importance of dynamic stiffness influence coefficients in treating free and forced vibration problems of beam structures has been demonstrated elsewhere [3–8].

To the authors knowledge, there is only one work in the literature, namely that of Koloušek [9], which deals with viscously damped frameworks with a continuous distribution of mass and different amounts of damping among the various structural members. However, that work deals only with the underdamped steady-state forced vibration case by employing a kind of dynamic stiffness influence coefficients in complex number form. Although only a certain class of continuous structures is considered in the present paper, namely that of beam structures, the results obtained are representative in that they demonstrate special features common to all continuous structures characterized by an infinite number of degrees of freedom. Other continuous structures for which dynamic stiffness influence coefficients can be constructed can also be studied by the proposed method. If this is not feasible, a finite element discretization and modeling of the structure as a discrete system with a finite number of degrees of freedom can be always done and the method of [2] then applied. Three examples dealing with axial, torsional and flexural vibrations are presented in detail in this paper to illustrate the proposed method and demonstrate the importance of critical damping surfaces.

2. FREE DAMPED VIBRATIONS OF A BEAM ELEMENT

In this section, dynamic stiffness influence coefficients for free axial, torsional and flexural vibratory motions of a damped linear elastic uniform beam element are defined and constructed. Either internal viscoelastic or external viscous damping is assumed. Internal viscoelastic damping is accounted for by assuming, for reasons of simplicity, that the beam material is a Kelvin solid, i.e. with a one-dimensional constitutive equation of the form

$$\sigma = H(1 + g\, \mathrm{d}/\mathrm{d}t)\epsilon, \tag{1}$$

where σ is the stress, ϵ is the strain, H stands for the modulus of elasticity E or the shear modulus G, g is the damping coefficient and t represents time. More general viscoelastic models as described, e.g. in [10] could also have been used without any particular difficulty. Equation (1) indicates that, under one dimensional states of stress, the formulation of the damped beam problem can be obtained from that of the corresponding undamped one by simply replacing H by H $(1 + g\, \mathrm{d}/\mathrm{d}t)$. When external viscous damping is present, it is accounted for in the displacement equation of the beam motion by a damping force per unit length P proportional to the velocity and opposing the motion, i.e.

$$P = -f(\mathrm{d}v/\mathrm{d}t), \tag{2}$$

where f is the coefficient of damping and v is the beam displacement.

Consider a uniform linear elastic beam element 1–2 of length L (Fig. 1) undergoing axial, torsional and flexural free damped motions which, on the basis of a strength-of-materials theory, are governed, respectively, by the following uncoupled equations:

$$\begin{gathered} EAu'' + gEA\dot{u}'' - m\ddot{u} = 0, \\ (AC/J)\phi'' + g(AC/J)\dot{\phi}'' - m\ddot{\phi} = 0, \\ EIv'''' + gEI\dot{v}'''' + m\ddot{v} = 0, \end{gathered} \tag{3}$$

for internal viscoelastic damping of the Kelvin type, and

$$\begin{gathered} EAu'' - f\dot{u} - m\ddot{u} = 0, \\ (AC/J)\phi'' - f\dot{\phi} - m\ddot{\phi} = 0, \\ EIv'''' + f\dot{v} + m\ddot{v} = 0, \end{gathered} \tag{4}$$

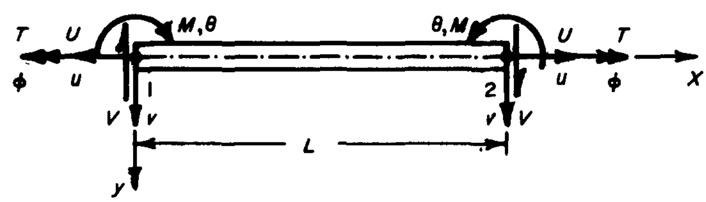

Fig. 1. Positive beam displacements and forces in mechanics convention.

for external viscous damping, where $u = u(x, t)$, $v = v(x, t)$ and $\phi = \phi(x, t)$ are the axial, lateral and angular displacements of the beam, respectively, E is the modulus of elasticity g and f are the coefficients of internal viscoelastic and external viscous damping, respectively, m is the mass per unit length of the beam, A, I, J and C are the area, moment of inertia, polar moment of inertia and torsional rigidity of the cross-section of the beam, respectively, primes indicate differentiation with respect to the distance x along the length of the beam and dots indicate differentiation with respect to the time t.

By assuming solutions of the form

$$\begin{aligned} u &= \bar{u}\,e^{\lambda t}, \\ \phi &= \bar{\phi}\,e^{\lambda t}, \\ v &= \bar{v}\,e^{\lambda t}, \end{aligned} \tag{5}$$

where $\bar{u}$, $\bar{\phi}$ and $\bar{v}$ are functions of x only and λ is, in general, a complex number, eqns (3) and (4) are reduced to

$$\begin{aligned} \bar{u}'' - K_1^2\bar{u} &= 0, \\ \bar{\phi}'' - K_2^2\bar{\phi} &= 0, \\ \bar{v}'''' + 4K^4\bar{v} &= 0, \end{aligned} \tag{6}$$

where

$$\begin{aligned} K_1^2 &= m\lambda^2/EA(1+g\lambda), \\ K_2^2 &= m\lambda^2/(AC/J)(1+g\lambda), \\ 4K^4 &= m\lambda^2/EI(1+g\lambda), \end{aligned} \tag{7}$$

for internal viscoelastic damping, and

$$\begin{aligned} K_1^2 &= (m\lambda^2+f\lambda)/AE, \\ K_2^2 &= (m\lambda^2+f\lambda)/(AC/J), \\ 4K^4 &= (m\lambda^2+f\lambda)/EI, \end{aligned} \tag{8}$$

for external viscous damping.

The general case of free damped motion is described by the solution of eqns (6), namely

$$\begin{aligned} \bar{u} &= D_1\,e^{K_1x} + F_1\,e^{-K_1x}, \\ \bar{\phi} &= D_2\,e^{K_2x} + F_2\,e^{-K_2x}, \\ \bar{v} &= e^{Kx}(D_3\cos Kx + F_3\sin Kx) + e^{-Kx}(D_4\cos Kx + F_4\sin Kx), \end{aligned} \tag{9}$$

where D_1, F_1, D_2, F_2, D_3, F_3, D_4 and F_4 are constants. Consider the beam element 1–2 of Fig. 2 with nodes 1 and 2, which has one, one and two degrees of freedom per node for axial, torsional and flexural motions, respectively. Figure 2 shows the positive directions of the nodal displacements and the corresponding nodal forces for the three kinds of motion. The dynamic stiffness influence coefficient Dij is defined as the force at the ith degree of freedom due to a displacement $1 \cdot e^{\lambda t}$ at the jth degree of freedom, while all the other displacements are zero. On the basis of this definition and by using the displacement functions (9) as well as the

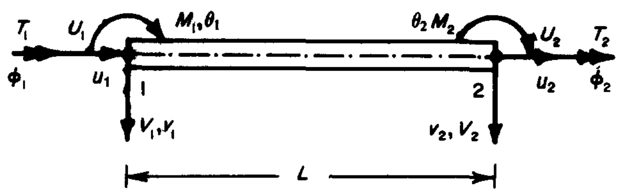

Fig. 2. Positive beam nodal displacements and forces.

force-displacement relations

$$\begin{aligned} U(x) &= AE\bar{u}'(x), \\ T(x) &= C\bar{\phi}'(x), \\ V(x) &= -EI\bar{v}'''(x), \\ M(x) &= -EI\bar{v}''(x), \end{aligned} \tag{10}$$

with positive directions as indicated in Fig. 1, one can construct the *Dij* coefficients for the three kinds of motion considered here. Thus, with the sign convention of Fig. 2, the following element nodal force-displacement relations in terms of the *Dij* coefficients result:

$$\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{bmatrix} D'_{11} & D'_{12} \\ D'_{21} & D'_{22} \end{bmatrix} \begin{Bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{Bmatrix}, \tag{11}$$

for the axial motion,

$$\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{bmatrix} D''_{11} & D''_{12} \\ D''_{21} & D''_{22} \end{bmatrix} \begin{Bmatrix} \bar{\phi}_1 \\ \bar{\phi}_2 \end{Bmatrix}, \tag{12}$$

for the torsional motion, and

$$\begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ D_{21} & D_{22} & D_{23} & D_{24} \\ D_{31} & D_{32} & D_{33} & D_{34} \\ D_{41} & D_{42} & D_{43} & D_{44} \end{bmatrix} \begin{Bmatrix} \bar{v}_1 \\ \bar{\theta}_1 \\ \bar{v}_2 \\ \bar{\theta}_2 \end{Bmatrix} \tag{13}$$

for the flexural motion, where

$$\begin{aligned} D'_{11} &= D'_{22} = AEK_1 \coth(K_1L), \\ D'_{12} &= D'_{21} = -AEK_1 \operatorname{cosech}(K_1L), \end{aligned} \tag{14}$$

$$\begin{aligned} D''_{11} &= D''_{22} = CK_2 \coth(K_2L), \\ D''_{12} &= D''_{21} = -CK_2 \operatorname{cosech}(K_2L), \end{aligned} \tag{15}$$

$$\begin{aligned} D_{11} &= D_{33} = 2NK^2(\gamma^2 + 4\gamma sc - 1), \\ D_{12} &= D_{21} = -D_{34} = -D_{43} = NK[\gamma^2 - 2\gamma(1 - 2s^2) + 1], \\ D_{13} &= D_{31} = (4K^2N/\sqrt{(\gamma)})[-\gamma^2(s^3 + sc^2 + c) + \gamma(-s^3 - sc^2 + c^2)], \\ D_{14} &= D_{41} = -D_{23} = -D_{32} = (2NK/\sqrt{(\gamma)})(s^3 + sc^2 + s)(\gamma^2 - \gamma), \\ D_{22} &= D_{44} = N(\gamma^2 - 4\gamma cs - 1), \\ D_{24} &= D_{42} = (2N/\sqrt{(\gamma)})[\gamma^2(s^3 + sc^2 - c) + \gamma(s^3 + sc^2 + c)], \\ N &= 2EIK/[\gamma^2 - 2\gamma(1 + 2s^2) + 1], \\ \gamma &= e^{2KL}, \quad s = \sin KL, \quad c = \cos KL, \end{aligned} \tag{16}$$

and where K_1, K_2 and K are given by (7) and (8) for internal and external damping, respectively.

3. CRITICAL DAMPING SURFACES

Consider a three-dimensional beam structure consisting of a finite number of uniform beam elements with a continuous distribution of mass and with different amounts of damping under free motion. The damping in every beam element may be internal viscoelastic or external viscous, or a combination of these and may be, in general, different for different kinds of free motion. The free motion of every element (and so that of the whole structure) is a combination of axial, torsional and flexural motion. Coupling effects between the various motions are neglected in this work.

The general equation of free motion for the above described beam structure is of the form (e.g. [7])

$$[D]\{x\} = [D]\{\psi\}e^{\lambda t} = \{0\}, \tag{17}$$

where $[D]$ is the structural dynamic stiffness matrix, whose elements are combinations of the dynamic stiffness influence coefficients Dij for axial, torsional and flexural motion given by (14)–(16), resulting from the superposition of the various element dynamic stiffness matrices and $\{x\}$ and $\{\psi\}$ are the vectors of the nodal structural displacements and displacement amplitudes, respectively. The matrix $[D]$ is thus a function of the damping properties of the various beam elements and the eigenvalues λ defined by eqns (5). Thus, in general,

$$[D] = [D(h_r, \lambda)], \quad (r = 1, 2, \dots q), \tag{18}$$

where h represents amounts of damping and q is an integer, in general, different from the order p of the matrix $[D]$.

Equation (17) has nontrivial solutions if, and only if,

$$\det[D(h_r, \lambda)] = |D(h_r, \lambda)| = 0. \tag{19}$$

The characteristic eqn (19) is a transcendental equation in λ which has an infinite number of roots corresponding to the infinite number of degrees of freedom of the structure under consideration. These roots can be negative real or complex, resulting in overdamped or critically damped aperiodically decaying free motion, or in underdamped oscillatory decaying free motion, respectively. If the system has no damping ($h_r = 0$), eqn (19) is satisfied for an infinite number of imaginary values of λ of the form $i\omega_0$, where ω_0 represents natural frequencies and $i = \sqrt{(-1)}$. If some (all) of the roots of (19) correspond to overdamping, underdamping or critical damping, the structure is called partially (completely) overdamped, underdamped or critically damped, respectively. Once the roots λ of (19) have been determined, one can solve (17) for the in general complex modal shapes $\{\psi\}$.

For overdamping or critical damping the roots of (19) are of the form

$$\lambda = -b, \quad (b > 0), \tag{20}$$

and (19) becomes

$$|D(h_r, -b)| = 0. \tag{21}$$

In the q-dimensional space with coordinates h_r ($r = 1, 2, \dots q$), eqn (21) represents a family of q-dimensional surfaces corresponding to overdamping or critical damping. The problem consists of determining that b which corresponds to the "critical damping surface." There are actually infinitely many "critical damping surfaces" since there are as many critical damping possibilities as the number of the roots λ. The general method for determining critical damping surfaces of linear discrete damped systems developed in [2], is extented here to linear continuous systems described by (17). Thus, having in mind that critical damping represents the threshold between overdamping and underdamping, one can conclude that among the S_b surfaces described by (21), the critical surface S_{cr} is the one for which the damping is a minimum, i.e.

$$(d/db)(|D(h_r, -b)|) = 0,$$
$$dh_r/db = 0, \quad r = 1, 2, \dots, q. \tag{22}$$

In principle, eqn (22) provides the b_{cr} as a function of the h_r's and thus the equation of critical damping surfaces is given by (21) with $b = b_{cr}$, i.e. by

$$|D(h_r, -b_{cr})| = 0. \tag{23}$$

In practice, however, one has to solve the system of simultaneous nonlinear equations (21) and (22) numerically in a manner analogous to that described in [2].

The above method for determining critical damping surfaces is quite general and applicable, in principle, to the most general space beam structure; however, the practical applicability of the method is limited to small order structures, because, to the authors knowledge, there is presently no efficient numerical method available for treating eqns (22). However, things are greatly simplified in the particular cases in which only axial or torsional or flexural free motion arise and one kind of damping (internal or external) is under consideration. Fortunately, these cases deal with very large classes of structures, such as plane frames or trusses which are usually considered to undergo only flexural free vibrations and simple and composite beams or shafts undergoing axial or torsional free vibrations, respectively.

Consider a beam structure undergoing only axial or torsional or flexural free vibrations under conditions of either internal or external damping. The coefficients K_{1r}, K_{2r} and K_r for axial, torsional and flexural free motion, respectively, of the rth beam element ($r = 1, 2, \ldots, \bar{q}, \bar{q} \leq q$) are given on account of (7) and (8) by

$$\begin{aligned} K_{1r}^2 &= [m_r/(EA)_r][\lambda^2/(1+g_r\lambda)], \\ K_{2r}^2 &= [m_r/(AC/J)_r](\lambda^2 + 2\beta_r\lambda), \\ 4K_r^4 &= [m_r/(EI)_r][\lambda^2/(1+g_r\lambda)], \end{aligned} \tag{24}$$

for internal viscoelastic damping, and

$$\begin{aligned} K_{1r}^2 &= [m_r/(EA)_r](\lambda^2 + 2\beta_r\lambda), \\ K_{2r}^2 &= [m_r/(AC/J_r](\lambda^2 + 2\beta_r\lambda), \\ 4K_r^4 &= [m_r/(EI)_r](\lambda^2 + 2\beta_r\lambda), \end{aligned} \tag{25}$$

for external viscous damping with

$$f_r = 2m_r\beta_r, \tag{26}$$

and where the coefficients g_r and β_r in (24) and (25) are, in general, different for different kinds of motion. For the undamped case of the rth beam element for which

$$\lambda_n = i\omega_{on}, \ (n = 1, 2, \ldots, \infty), \tag{27}$$

where ω_{on} represents the natural frequency in the nth mode of vibration and $i = \sqrt{(-1)}$, eqns (7) and (8) are replaced by

$$\begin{aligned} K_{1r0}^2 &= -\omega_{on}^2[m_r/(EA)_r], \\ K_{2r0}^2 &= -\omega_{on}^2[m_r/(AC/J)_r], \\ 4K_{ro}^4 &= -\omega_{on}^2[m_r/(EI)_r]. \end{aligned} \tag{28}$$

When one particular kind of free motion of the beam structure is considered, the structural dynamic matrix $[D]$ consists of linear combinations of Dij coefficients with the same type of K coefficients. In that case (22) becomes

$$\begin{aligned} (\mathrm{d}/\mathrm{d}b)(|D|) &= \sum_r (\mathrm{d}/\mathrm{d}K_r)(|D|) \cdot (\mathrm{d}K_r/\mathrm{d}b) = 0, \\ (\mathrm{d}h_r/\mathrm{d}b) &= 0, \quad r = 1, 2, \ldots, q, \end{aligned} \tag{29}$$

which can be recognized as direct extensions of the corresponding equations for discrete systems [2]. The derivatives $\mathrm{d}K_r/\mathrm{d}b$ for the various cases of motion and kinds of damping can be computed from (24) and (25) for $\lambda = -b$, and in conjunction with $(29)_2$ take the following forms:

$$\begin{aligned} \mathrm{d}K_{1r}/\mathrm{d}b &= [m_r/2K_{1r}(EA)_r b(2-g_rb)/(1-g_rb)^2, \\ \mathrm{d}K_{2r}/\mathrm{d}b &= [m_r/2K_{2r}(AC/J)_r]b(2-g_rb)/(1-g_rb)^2, \\ \mathrm{d}K_r/\mathrm{d}b &= [m_r/16K_r^3(EI)_r]b(2-g_rb)/(1-g_rb)^2, \end{aligned} \tag{30}$$

for internal viscoelastic damping, and

$$\begin{aligned} \mathrm{d}K_{1r}/\mathrm{d}b &= [m_r/K_{1r}(EA)_r](b-\beta_r), \\ \mathrm{d}K_{2r}/\mathrm{d}b &= [m_r/K_{2r}(AC/J)_r](b-\beta_r), \\ \mathrm{d}K_r/\mathrm{d}b &= [m_r/8K_r^3(EI)_r](b-\beta_r), \end{aligned} \tag{31}$$

for external viscous damping.

The particular point $h_1 = h_2 = \ldots = h_q = h$ of a critical damping surface for which conditions (29) hold true can be easily determined. Thus, on account of (30) and (31) with $\beta_r = \beta$ and $g_r = g$, one can easily see that $(29)_1$ is satisfied for

$$2 - gb = 0, \tag{32}$$

for internal viscoelastic damping and

$$b - \beta = 0, \tag{33}$$

for external viscous damping. Substitution of (32) and (33) into (24) and (25) in conjunction with $\lambda = -b$, $g_r = g$, and $\beta_r = \beta$ and comparison of the results with (28) leads to a critical value b_{cr} of b given by

$$b_{ncr} = \omega_{on}, \quad n = 1, 2, \ldots, \infty, \tag{34}$$

for all kinds of motions and damping cases. Thus, (23) takes the form

$$|D(h_n, -\omega_{on})| = 0 \tag{35}$$

and serves to determine the value of $h = h_n$ for every n numerically. Notice that in this case the structure possesses classical normal modes. Equation (35) also indicates that a beam structure with a continuous distribution of mass, one kind of damping and under one kind of free motion has an infinite number of critical damping surfaces and consequently complete critical damping or overdamping are achieved for $n \to \infty$.

The above results could have been obtained in a different way as follows: When $h_1 = h_2 = \ldots = h_q = h$, combination of (24), (25) and (28) leads to the conclusion that the value of λ_n in any mode n will be such that

$$\lambda_n^2/(1 + g\lambda_n) = -\omega_{on}^2 \tag{36}$$

for internal viscoelastic damping and

$$\lambda_n^2 + 2\beta\lambda_n = -\omega_{on}^2, \tag{37}$$

for external viscous damping. Under conditions of underdamping

$$\lambda_n = -b_n \pm i\omega_n, \tag{38}$$

where ω_n represents the damped natural frequency in the nth mode and by combining (38) with (36) and (37) one obtains the relations

$$b_n = (1/2)g\omega_{on}^2, \quad \omega_n = (1/2)\omega_{on}(4 - g^2\omega_{on}^2)^{1/2} \tag{39}$$

for internal viscoelastic damping, and

$$b_n = \beta, \quad \omega_n = (\omega_{on}^2 - \beta^2)^{1/2} \tag{40}$$

for external viscous damping. Considering critical damping as the threshold between under-damping and overdamping, one can imagine the state of critical damping as the limit of the underdamping state as the amount of damping increases so that ω_n approaches zero. Thus, the conditions of critical damping, in view of (39) and (40), are

$$g = 2/b_{ncr}, \quad b_{ncr} = \omega_{on} \tag{41}$$

for internal viscoelastic damping, and

$$\beta = b_{ncr}, \quad b_{ncr} = \omega_{on}, \tag{42}$$

for external viscous damping. These equations are identical with (32)–(34).

Three illustrative examples are given below.

4. EXAMPLES

Example 1

Consider a uniform cantilever beam of length L, mass per unit length m and axial rigidity AE. In this case, damping is uniform; the example is included simply to illustrate the proposed method in a classical case. The equation of free axial vibration of this beam, obtained from (11) for $U_1 = U_2 = 0$ and $\bar{u}_1 = 0$ reads

$$D'_{22}\bar{u}_2 = 0. \tag{43}$$

Thus, eqn (21), on account of (14), $(24)_1$, $(25)_1$ and (43), becomes

$$\coth(K_1L) = 0 \tag{44}$$

where

$$K_1^2 = (m/AE)(b^2/(1 + gb)] \tag{45}$$

for internal viscoelastic damping, and

$$K_1^2 = (m/AE)(b^2 - 2\beta b) \tag{46}$$

for external viscous damping. The equation of critical damping surfaces is again given by (44) but with (45) and (46) being replaced by

$$K_1^2 = (m/AE)[\omega_{on}^2/(1 + g\omega_{on})], \tag{47}$$

$$K_1^2 = (m/AE)(\omega_{on}^2 - 2\beta\omega_{on}), \tag{48}$$

for internal and external damping, respectively, in view of the condition (34) for critical damping. The natural frequencies ω_{on} can be obtained from (44) with K_1 given by $(28)_1$ and are of the form

$$\omega_{on} = (1/2L)(2n - 1)\pi\sqrt{(AE/m)}, \quad n = 1, 2, \ldots, \infty, \tag{49}$$

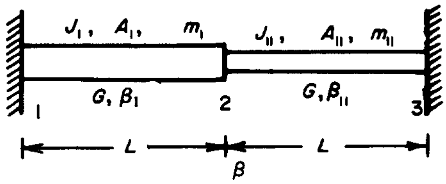

Fig. 3. Composite shaft in torsional free motion for Example 2.

Thus, for both kinds of damping, the critical damping surfaces are actually an infinite set of points which, on account of (32)–(34), are given by

$$g_n = 2/\omega_{on} \tag{50}$$

for internal viscoelastic damping, and

$$\beta_n = \omega_{on} \tag{51}$$

for external viscous damping.

Example 2

Consider the composite shaft 1–2–3 of Fig. 3 which consists of two uniform beams of circular cross sections with length L and shear modulus G. The polar moment of inertia, the cross-sectional area, the mass per unit of length and the external viscous damping coefficient are symbolized by J_I, A_I, m_I, β_I and J_{II}, A_{II}, m_{II}, β_{II} for the beams (1–2) = I and (2–3) = II, respectively. The equation of free torsional motion of this shaft is obtained with the help of (12) by superimposing the dynamic stiffnesses of the two beam elements I and II and applying the boundary conditions $\bar{\phi}_1 = \bar{\phi}_3 = 0$. Thus, eqn (21) becomes

$$[(D''_{22})_I + (D''_{11})_{II}] = 0 \tag{52}$$

where, from (15) for a circular cross section ($C = GJ$),

$$(D''_{22})_I = GJ_I K_{2I} \coth(K_{2I}L), \qquad (D''_{11})_{II} = GJ_{II} K_{2II} \coth(K_{2II}L), \tag{53}$$

with K_{2I} and K_{2II} given by $(27)_2$ with $\lambda = -b$ and $C = GJ$. Equation (29), with the aid of (52), (53) and $(31)_2$ with $C = GJ$, yields

$$(\mathrm{d}(D''_{22})_I/\mathrm{d}K_{2I})(\mathrm{d}K_{2I}/\mathrm{d}b) + (\mathrm{d}(D''_{11})_{II}/\mathrm{d}K_{2II})(\mathrm{d}K_{2II}/\mathrm{d}b) = 0, \tag{54}$$

where

$$\begin{aligned} \mathrm{d}(D''_{22})_I/\mathrm{d}K_{2I} &= GJ_I[\coth(K_{2I}L) - K_{2I}L \operatorname{cosch}^2(K_{2I}L)], \\ \mathrm{d}(D''_{11})_{II}/DK_{2II} &= GJ_{II}[\coth(K_{2II}L) - K_{2II}L \operatorname{cosch}^2(K_{2II}L)], \\ \mathrm{d}K_{2i}/\mathrm{d}b &= (m_i/K_{2i}A_iG)(b - \beta_i), \quad i = I, II. \end{aligned} \tag{55}$$

The natural frequencies of the system can be obtained from (52) with

$$K_{2I}^2 = K_{2II}^2 = \bar{K}^2 = -\omega_{on}^2(\rho/G), \tag{56}$$

where

$$\rho = (m_I/A_I) = (m_{II}/A_{II}), \tag{57}$$

represents the mass density of the shaft material. Use of (56) and (57) reduces (52) to

$$(J_I + J_{II}) \cot(\bar{K}L) = 0 \tag{58}$$

with solution

$$\omega_{on} = (1/2L)(2n-1)\pi\sqrt{(G/\rho)},\ n = 1, 2, \ldots, \infty. \tag{59}$$

The critical damping surfaces are described by (52) with b_{cr} obtained as a function of β_I and β_{II} from (54) and (55) and form an infinite set of curves in the $\beta_I - \beta_{II}$ plane. For the particular case of $\beta_{II} = 0$, $J_I/J_{II} = 2$, $G/\rho = 1$ and $L = 1$, a computer solution of the simultaneous equations (52), (54) and (55) provides the following values for the first three roots β_I:

$$\beta_I = 2.03, 5.16, 8.31. \tag{60}$$

These values correspond to the first three natural frequencies

$$\omega_{o1} = \pi/2 = 1.571,\quad \omega_{o2} = 3\pi/2 = 4.712,\quad \omega_{o3} = 5\pi/2 = 7.854, \tag{61}$$

and indicate that the critical damping surfaces are just an infinite set of points along the β_I axis.

Example 3

Consider the uniform continuous beam 1–2–3 of Fig. 4 with bending rigidity EI and mass per unit of length m. The structure consists of two beam elements of the same length L but with different external viscous damping coefficients β_{I} and β_{II} as shown in Fig. 4. The equation of free flexural motion of this continuous beam is obtained with the aid of (13) by superimposing the dynamic stiffnesses of the two elements (1–2) = I and (2–3) = II and applying the boundary conditions $\bar{v}_1 = \bar{\theta}_1 = \bar{v}_2 = \bar{v}_3 = \bar{\theta}_3 = 0$. Thus, eqn (21) becomes

$$[(D_{44})_I + (D_{22})_{II}] = 0, \tag{62}$$

where $(D_{44})_I = (D_{22})_I$ and $(D_{22})_{II}$ are furnished by (16) with K_I and K_{II} given by $(25)_3$ with $\lambda = -b$.

Equation (29), with the aid of (62), $(31)_3$ and (16), yields

$$\begin{gathered}(\mathrm{d}(D_{22})_I/\mathrm{d}K_I)(\mathrm{d}K_I/\mathrm{d}b) + (\mathrm{d}(D_{22})_{II}/\mathrm{d}K_{II})(\mathrm{d}K_{II}/\mathrm{d}b) = 0\\ \mathrm{d}K_i/\mathrm{d}b = (m/8K_i^3EI)(b-\beta_i),\\ \mathrm{d}(D_{22})_i/\mathrm{d}K_i = \{[2(\gamma^2 - 4\gamma cs - 1) + 8K_i(\gamma^2 - 2\gamma cs - \gamma c^2 + \gamma s^2)]\\ \cdot[\gamma^2 - 2\gamma(1+2s^2)+1] - 8K_i(\gamma^2 - 4\gamma cs - 1)(\gamma^2 - \gamma - 2\gamma s^2 - 2\gamma cs)\}\\ /[\gamma^2 - 2\gamma(1+2s^2)+1]^2,\\ K_i = (m/\sqrt{(2)}EI)(b^2 - 2b\beta_i)^{1/4},\quad i = I, II.\end{gathered} \tag{63}$$

The natural frequencies of the system can be obtained from (62) with

$$4\mathrm{K}_{\mathrm{I}}^4 = 4\mathrm{K}_{\mathrm{II}}^4 = -(\mathrm{m}/\mathrm{EI})\omega_{\mathrm{on}}^2, \tag{64}$$

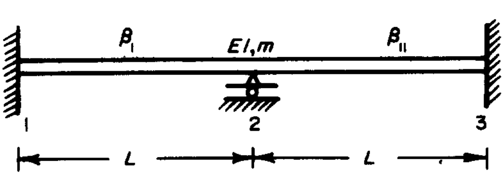

Fig. 4. Continuous beam in flexural free motion for Example 3.

or directly from

$$2D_{22}=0, \tag{65}$$

where the dynamic stiffness coefficient without damping D_{22} is taken from Ref. [4 or 7] and is of the form

$$D_{22}=EIK(\sin KL\cosh KL-\cos KL\sinh KL)/(1-\cos KL\cosh KL) \tag{66}$$

with

$$K^2=\omega_{on}\sqrt{(m/EI)}. \tag{67}$$

Use of (66) reduces (65) to

$$\tan KL-\tanh KL=0, \tag{68}$$

which has an infinite number of roots, the first three of which are given to third decimal accuracy by

$$KL=3.927,\ 7.069,\ 10.210. \tag{69}$$

Thus, by combining (67) and (69), the following expressions for the first three natural frequencies are obtained:

$$\omega_{on}=(\mu_n^2/L^2)\sqrt{(EI/m)},\quad n=1,2,3,$$
$$\mu_1^2=15.421,\quad \mu_2^2=49.971,\quad \mu_3^2=104.244. \tag{70}$$

The critical damping surfaces are described by (62) with b_{cr} obtained as a function of β_I and β_{II} from (63) and form an infinite set of nonlinear curves in the $\beta_I-\beta_{II}$ plane. All these curves are symmetric about the line $\beta_I=\beta_{II}$, as it can be very easily seen from (16) and $(25)_3$ with $\lambda=-b$ that an interchange of β_I and β_{II} leaves (62) unaffected; this was expected in view of the symmetry of the structure. For the particular points of the curves for which $\beta_I=\beta_{II}=\beta$, (42) yields

$$\beta_n=b_{ncr}=\omega_{on}. \tag{71}$$

Figure 5 shows the first three critical damping curves corresponding to the first three frequencies of (70). These curves were constructed by numerically solving the system of simultaneous equations (62) and (63) for $EI=L=m=1$; the computations were done by computer using complex arithmetic due to the fact that K_I and K_{II} in $(63)_4$ are, in general, complex numbers. It was observed during the computation that the values of b_{cr} for every curve were very close to the value of b_{cr} corresponding to the point $\beta_I=\beta_{II}$, i.e., to the value $\beta_{ncr}=\omega_{on}$. This suggested an approximate construction of the curves on the basis of (62) with

$$4K_i^4=(m/EI)(\omega_{on}^2-2\beta_i\omega_{on}) \tag{72}$$
$$i=I,\ II,\ n=1,2,\ldots,\infty.$$

These approximate curves drawn in Fig. 5 for $EI=L=m=1$ practically coincide with the "exact" curves. Even though this suggests a simple way to construct very good approximate curves, no generalizations beyond the above observations are presently available. The critical damping curves $C_1, C_2, \ldots, C_\infty$ of the structure of Fig. 4 with the exception of the C_∞ are all partially critical and separate the $\beta_I-\beta_{II}$ plane in an infinite number of regions $R_1, R_2 \ldots, R_\infty$ of which only the first four are shown in Fig. 5. Region R_1 represents complete underdamping, while complete overdamping is achieved only at infinity as all the other regions represent partial overdamping.

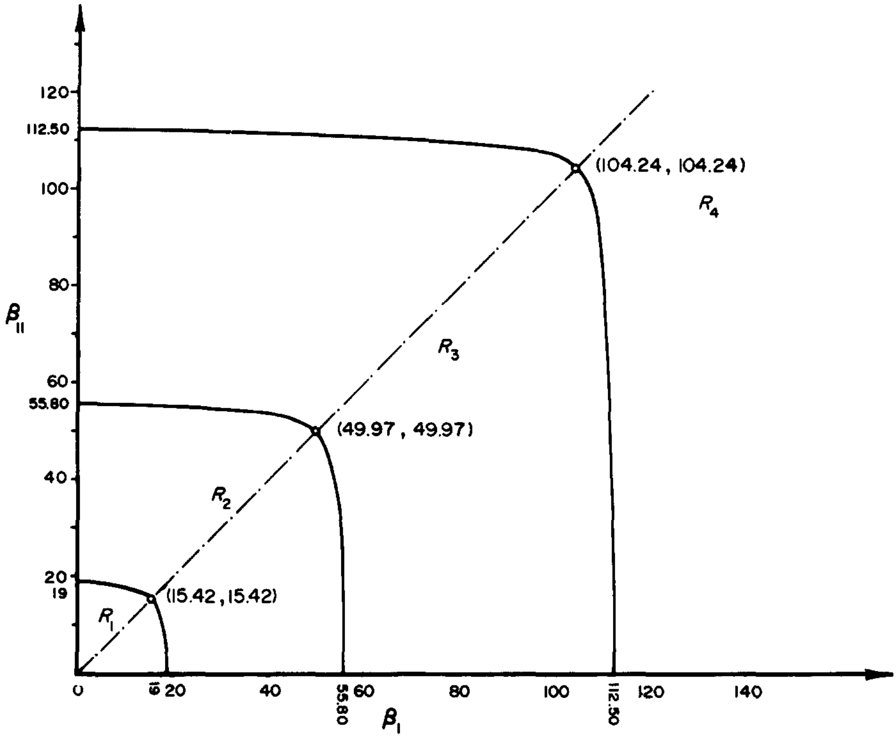

Fig. 5. The first three critical damping curves of the continuous beam of Example 3.

The damping curves of Fig. 5 are the "exact" damping curves of the continuous beam as computed with the aid of the dynamic stiffness influence coefficients constructed on the basis of the exact equations of motion. It is interesting to study the degree of approximation obtained by constructing approximate critical damping curves of the continuous beam, on the assumption that the structure is discretized by lumping the mass as shown in the alternative models of Fig. 6. The equation of free flexural motion of the two degrees of freedom model of Fig. 6(a), on the assumption that its viscous damping matrix is diagonal and by eliminating rotational degrees of freedom, takes the form

$$\begin{bmatrix} M' & 0 \\ 0 & M' \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \frac{12EI}{7(l')^3}\begin{bmatrix} 11 & 3 \\ 3 & 11 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{73}$$

where x_1, c_1 and x_2, c_2 represent vertical deflections and damping coefficients for the points 1 and 2, respectively. The static stiffness coefficients for a beam element were taken from [11]. The natural frequencies ω_0 of the two degrees of freedom system are the roots of the equation

$$\begin{vmatrix} -\omega_0^2M' + 11\xi & 3\xi \\ 3\xi & -\omega_0^2M' + 11\xi \end{vmatrix} = 0 \tag{74}$$

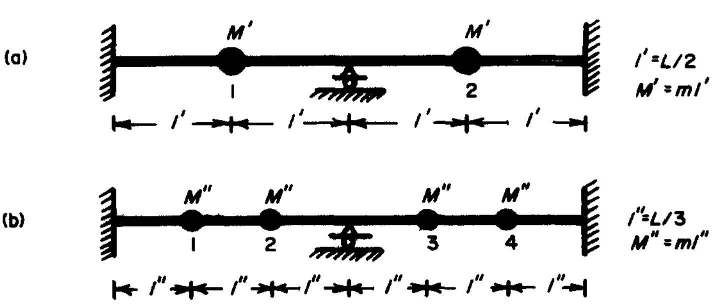

Fig. 6. Lumped-mass discretizations of the continuous beam of Example 3.

where

$$\xi = 12EI/7(l')^3. \tag{75}$$

From the approximate method of [2], the equations of the critical damping curves of the system are

$$\left|\omega_{on}^2 M'\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \omega_{on}2M'\begin{bmatrix}\beta_1 & 0\\ 0 & \beta_2\end{bmatrix} + \xi\begin{bmatrix}11 & 3\\ 3 & 11\end{bmatrix}\right| = 0 \tag{76}$$

where

$$c_1 = 2M'\beta_1, \quad c_2 = 2M'\beta_2, \quad n = 1, 2.$$

For the case of $EI = L = m = 1$,

$$M' = 0.5, \quad \xi = 13.714, \tag{77}$$

and the solution of (74) is

$$\omega_{o1} = 14.813, \quad \omega_{o2} = 19.596 \tag{78}$$

forming a lower bound to the exact solution (70). A similar computation for the four degrees of freedom model of Fig. 6(b) provides the following values for the natural frequencies:

$$\omega_{o1} = 15.357, \quad \omega_{o2} = 22.055, \quad \omega_{o3} = 45.656, \quad \omega_{o4} = 51.254. \tag{79}$$

It is obvious from (72), (78) and (79) that both of the discrete systems provide an acceptable approximation for the first natural frequency only. Consequently, both of the discrete systems can provide a good approximation of only the first critical damping curve. Figure 7 shows the first critical damping curve for the continuous beam as obtained by the continuous as well as the two discrete models. The main conclusion of this comparison study is that for discrete lumped-mass models only the first few critical damping surfaces are close to the exact ones. Accuracy of representation of the higher damping surfaces increases by increasing the order of the discrete model.

5. CONCLUSIONS

On the basis of the preceding discussion, the following conclusions can be drawn with regard to the free motion of linear structures composed of beams with a continuous distribution of mass and different amounts of internal and/or external viscous damping:

(1) There exist infinitely many "critical damping surfaces" for every structure which represent

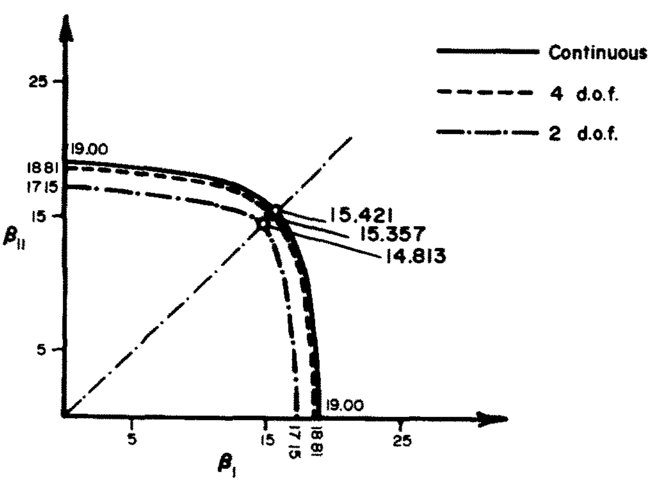

Fig. 7. Approximations of the first critical damping curve of the structure of Example 3.

the loci of combinations of damping leading to partial or complete critical damped motion and thus separating regions of partial or complete underdamping from those of overdamping. The dimension of these surfaces is equal to the number of independent amounts of damping present in the system.

(2) A general method is proposed for determining the equations of these critical damping surfaces in conjunction with dynamic stiffness influence coefficients which are functions of the damping and inertia properties of the structure. This method is greatly simplified for systems which undergo only one kind of motion, namely axial or torsional or flexural. For these systems there are infinitely many partially critical damping surfaces creating one region of complete underdamping and infinitely many regions of partial overdamping as complete critical damping or overdamping are achieved only for infinite damping values.

(3) The critical damping surfaces obtained by the proposed method represent the exact solution of the problem since they are constructed with the aid of dynamic stiffness influence coefficients based on the exact solution of the equation of motion. For complicated beam structures or other structures with continuous distribution of stiffness, mass and damping properties, such as discs, plates and shells, a discretization of the structure and application of the methods of Ref. [2] is probably the most practical approach. The resulting damping surfaces will of course be approximate, with a degree of accuracy decreasing as surfaces corresponding to higher frequencies are considered.

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REFERENCES

1. T. K. Caughey and M. E. J. O'Kelly, Classical normal modes in damped linear dynamic systems *J. Appl. Mech.* **32**, 583–588 (1965).
2. D. E. Beskos and B. A. Boley, Critical damping in linear discrete dynamic systems. *J. Appl Mech.* **47**, 627–630 (1980).
3. R. W. Clough, and J. Penzien, *Dynamics of Structures*. McGraw-Hill, New York (1975).
4. D. E. Beskos, Use of Dynamic Influence Coefficients in Vibration Problems, Ph.D. Thesis. Cornell University, Ithaca, New York (1974).
5. D. E. Beskos and B. A. Boley, Use of dynamic influence coefficients in forced vibration problems with the aid of Laplace transform. *Comp. Struct.* **5**, 263–269 (1975).
6. G. V. Narayanan and D. E. Beskos, Use of dynamic influence coefficients in forced vibration problems with the aid of fast Fourier transform. *Comp. Struct.* **9**, 145–150 (1978).
7. D. E. Beskos, Dynamics and stability of plane trusses with gusset plates. *Comp. Struct.* **10**, 785–795 (1979).
8. G. D. Manolis and D. E. Beskos, Thermally induced vibrations of beam structures. *Comp. Meth. Appl. Mech. Engng.* **21**, 337–355 (1980).
9. V. Koloušek, *Dynamics in Engineering Structures*. Butterworths, London (1973).
10. B. A. Boley and J. H. Weiner, *Theory of Thermal Stresses*. Wiley, New York (1960).
11. H. C. Martin, *Introduction to Matrix Methods of Structural Analysis*, McGraw-Hill, New York (1966).